Thankful Quiz

① What is your favorite movie?

② Finish the definition.

ⓐ The sequence (x_n) is <u>bounded</u> if .... $\exists M>0$ s.t. $|x_n| \le M \; \forall n \in \mathbb{N}$.

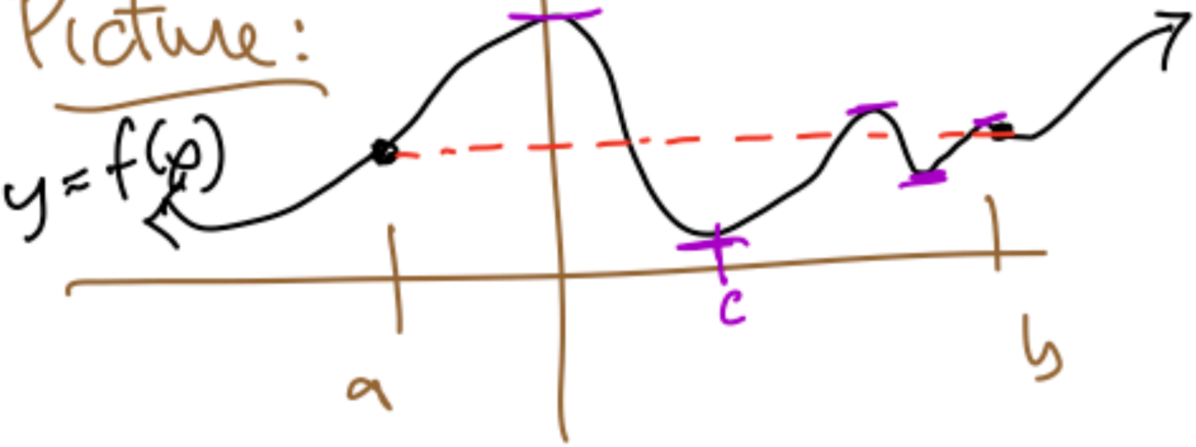
ⓑ $f: A \to \mathbb{R}$ is uniformly continuous if $\forall \varepsilon > 0$ ... $\exists \delta > 0$ s.t. if $|x-c|<\delta$ and $x, c \in A$, then $|f(x)-f(c)|<\varepsilon$.

ⓒ If I is an interval and $f: I \to \mathbb{R}$ is a function and $c \in I$, then $f'(c) = \lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c}$.

Last time:

Rolle's Thm. If $f: [a,b] \to \mathbb{R}$ is continuous & diff'ble on (a,b), and if $f(a) = f(b)$, then $\exists c \in (a,b)$ where $f'(c) = 0$.

Picture:

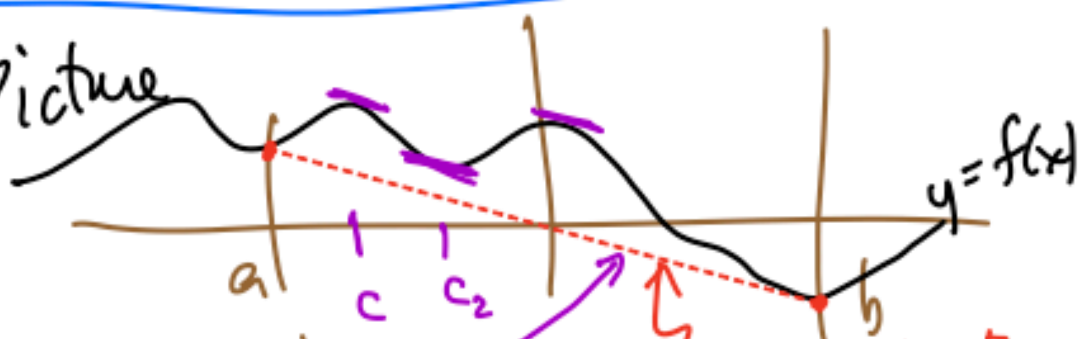


Mean Value Theorem (MVT)

Let $f:[a,b]\to\mathbb{R}$ be continuous & diff'ble on (a,b).
Then $\exists\, c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Picture



slope $=\dfrac{f(b)-f(a)}{b-a}$

equation

$y-y_0=m(x-x_0)$ $\quad y-f(a)=\dfrac{f(b)-f(a)}{b-a}(x-a)$

Pf. Let $g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]$.

Then g is continuous (ACT) on $[a,b]$ and diff'ble on (a,b), by the ADT.

And
$$g(a) = f(a) - \left[f(a) + \frac{f(b)-f(a)}{b-a}\underbrace{(a-a)}_{0}\right]$$
$$= 0;$$
$$g(b) = f(b) - \left[\underbrace{f(a) + \frac{f(b)-f(a)}{b-a}(b-a)}_{f(b)}\right]$$
$$= 0$$

Thus, by Rolle's Thm, $\exists c \in (a,b)$ such that

$g'(c) = 0$, which by ADT

is $g'(c) = f'(c) - \left(\frac{f(b)-f(a)}{b-a}\right)$.

Thus $f'(c) = \frac{f(b)-f(a)}{b-a}$. $\blacksquare$

<u>Corollary.</u> Let f be cont. on $[a,b]$ and diff'ble on (a,b), and suppose $f'(x)=0\ \forall x\in(a,b)$. Then $f(x)$ is constant on $[a,b]$.

<u>Pf.</u> Let $x\in(a,b]$. Then by MVT on $[a,x]$, $\exists c\in(a,x)$ s.t.

$$f'(c)=\frac{f(x)-f(a)}{x-a}.$$

But, then $0=\dfrac{f(x)-f(a)}{x-a}$, so $f(x)=f(a)$.

Thus, f is constant on $[a,b]$. ▨

<u>More (interesting facts):</u>

① Generalized MVT:

If f,g satisfy the hypothesis of the MVT and $g(a)\neq g(b)$, then

$\exists c \in (a,b)$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

② Interesting Example – Weierstrass function

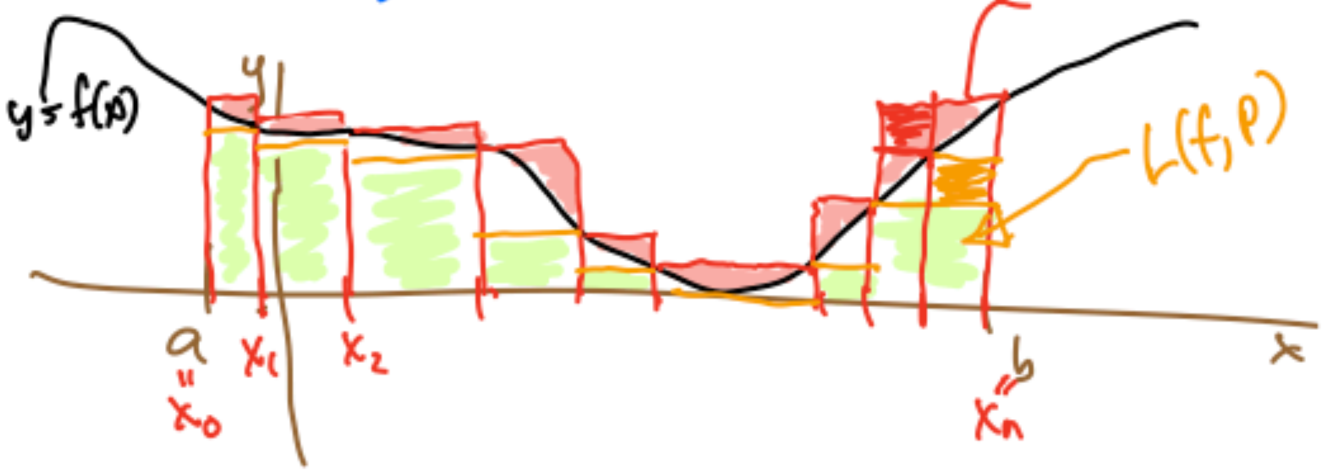
Let $W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x)$.

(see Desmos pictures with different $0<a<1$, $b>1$)

$\rightarrow$ $W(x)$ continuous, but if b is large $\left(> \frac{1}{a}\right)$, then $W'(x)$ does not exist at any $x \in \mathbb{R}$.

Integration

<u>Idea</u> — Riemann integrable functions — functions that you can integrate.



$P = \{x_0, x_1, \ldots, x_n\}$ a partition $\qquad x_j < x_{j+1}$ for $0 \le j \le n-1$

Want $U(f,P) - L(f,P)$ to be small as the partitions P get more fine.

In fact we want

$$U(f,P) - L(f,P) \to 0.$$

$\int_a^b f$

Rigorous Definitions

If $[a,b]$ is an interval, a set $P = \{a = x_0, x_1, \dots, x_n = b\}$ with

$$x_0 < x_1 < \dots < x_n$$

is called a <u>Partition</u> of $[a,b]$.

Given a bounded fcn $f : [a,b] \to \mathbb{R}$, and P as above, we define

$$U(f,P) = \sum_{k=1}^{n} M_k \Delta x_k$$

$$L(f,P) = \sum_{k=1}^{n} m_k \Delta x_k,$$

where $\Delta x_k = x_k - x_{k-1}$,

$$M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\}$$

$$m_k = \inf\{f(x) : x_{k-1} \le x \le x_k\}.$$

$U(f) := \inf\{U(f,P) : P \text{ is a partition of } [a,b].\}$

$L(f) := \sup\{L(f,P) : P \text{ is a partition of } [a,b].\}.$

<u>Defn.</u> We say that the bounded function f is <u>Riemann integrable</u> on $[a,b]$ if $U(f) = L(f)$. In this case, we say $\int_a^b f = U(f) = L(f)$.

<u>Lemmas:</u>

① If $P_1 \subseteq P_2$ for partitions P_1 & P_2 of $[a,b]$, we say P_2 is a refinement of P_1, and

$$U(f,P_1) \geq U(f,P_2)$$
$$L(f,P_1) \leq L(f,P_2).$$

② For any two partitions P_3, P_4 of $[a,b]$, we always have

$$U(f,P_3) \geq U(f) \underset{\text{= if } f \text{ is integrable.}}{\geq} L(f) \geq L(f,P_4).$$

Equivalent Defns of Riemann integrable:

- ε - Criterion for "Riemann integrable"

f is Riem. Integrable on $[a,b]$

$\iff \forall \varepsilon > 0$, $\exists$ partition P_ε of $[a,b]$ such that

$$\underbrace{U(f,P_\varepsilon) - L(f,P_\varepsilon)}_{\sum_{k=1}^{n} (M_k - m_k)\Delta x_k} < \varepsilon$$

- sequential criterion for Riem. int.

f is Riem integrable on $[a,b]$

$\iff$ $\exists$ a sequence (P_n) of partitions of $[a,b]$ such that $\lim_{n\to\infty} U(f,P_n) - L(f,P_n) = 0$.

$\llcorner$ if so, $\int_a^b f = \lim_{n\to\infty} U(f,P_n)$

$= \lim_{n\to\infty} L(f,P_n)$.